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INTERPOLATING VARIETIES FOR WEIGHTED SPACES OF ENTIRE FUNCTIONS IN $\mathbf{C}^n$

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Abstract

We prove in this paper that a given discrete variety V in $\mathbf{C}^n$ is an interpolating variety for a weight p if and only if V is a subset of the variety $\{\xi \in \mathbf{C}^n : f_1(\xi) = f_2(\xi) = \cdots = f_n(\xi) = 0\}$ of m functions $f_1, \cdots, f_m$ in the weighted space the sum of whose directional derivatives in absolute value is not less than $\epsilon \exp(-Cp(\zeta))$, $\zeta \in V$ for some constants ϵ, $C > 0$. The necessary and sufficient conditions will be also given in terms of the Jacobian matrix of $f, \cdots, f_m$. As a corollary, we solve an open problem posed by Berenstein and Taylor about interpolation for discrete varieties.

1. Introduction. In this note, we shall study the problem of finding necessary and sufficient conditions for a given discrete subset of $\mathbf{C}^n$ to be interpolating for spaces of entire functions in several complex variables satisfying growth conditions.

Let $V = \{\zeta_k\}$ be a discrete subset of $\mathbf{C}^n$ and $p(\xi)$, $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{C}^n$, a plurisubharmonic weight function on $\mathbf{C}^n$. We consider the following interpolation problem: under what conditions is it true that for any sequence $\{a_k\}$ of complex numbers satisfying the growth condition

$$|a_k| \leq A \exp(Bp(\zeta_k)), \quad k \in \mathbf{N} := \{1, 2, \cdots\}$$

for some constants A, $B > 0$, there exists an entire function f on $\mathbf{C}^n$ such that $f(\zeta_k) = a_k$ and f satisfies the same kind of estimate for $\xi \in \mathbf{C}^n$, namely, $f \in A_p(\mathbf{C}^n)$ or equivalently

$$|f(\xi)| \leq A' \exp(B'p(\xi))$$

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for some constants A', $B' > 0$. We will then say that V is an interpolating variety (for the weight p or the space $A_p(\mathbf{C}^n)$).

In the case when V is a complete intersection and defined by so-called slowly decreasing functions, some interpolation results have been known in [**BT2**], where a vector function $F = (F_1, F_2, \cdots, F_n)$, $F_i \in A_p(\mathbf{C}^n)$, is called slowly decreasing if and only if there exist ϵ, C, C_1, $C_2 > 0$ such that

(i) the connected components of the set $S(F; \epsilon, C)$ are bounded, where $S(F; \epsilon, C) = \{\xi \in \mathbf{C}^n : |F(\xi)| = (\sum_{j=1}^n |F_j(\xi)|^2)^{1/2} < \epsilon \exp(-Cp(\xi))\}$; and

(ii) if Ω is a component of $S(F; \epsilon, C)$, then $p(\xi) \leq C_1 p(w) + C_2$, for all ξ, $w \in \Omega$.

Note that the definition depends on the weight function p. F may be slowly decreasing for one weight and not for another. We refer to [**BT2**] for a class of examples of slowly decreasing functions. With this notion, the interpolation theorem in [**BT2**] about discrete varieties can be stated as follows:

Let $F = (F_1, F_2, \cdots, F_n)$, $F_i \in A_p(\mathbf{C}^n)$, be slowly decreasing. Assume further that the points $\{\zeta_k\}$ of $F(\xi) = 0$ are simple; that is,

$$\det J_F(\zeta_k) \neq 0 \quad (J_F(\xi) = \text{Jacobian matrix of } F).$$

Then $V = \{\zeta_k\}$ is interpolating for $A_p(\mathbf{C}^n)$ if and only if there exist constants $\epsilon > 0$, $C > 0$ such that

$$|\det J_F(\zeta_k)| \geq \epsilon \exp(-Cp(\zeta_k)), \quad k \in \mathbf{N}.$$

For $n = 1$, $p(z) = p(|z|)$, this result is due to A. F. Leont'ev [**L**]. For $n = 1$, $p(z) = |\operatorname{Im} z| + \log(1 + |z|)$, this result is due to Ehrenpreis and Malliavin [**EM**]. For $n = 1$ and general weights p, this result is given in [**BT1**].

Note that the above result only applies to the case of discrete varieties $V = \{\zeta_k\}$ which are exactly complete intersections of n slowly decreasing functions $F_1, F_2, \cdots, F_n$, i.e.,

$$V = Z(F_1, F_2, \cdots, F_n) := \{\xi \in \mathbf{C}^n : F_1(\xi) = F_2(\xi) = \cdots = F_n(\xi)\}$$

with $F = (F_1, F_2, \cdots, F_n)$ slowly decreasing. It was pointed out in [**BT2**, p. 210] that the case when the variety is not such a complete intersection seems to be quite different. Therefore it is a natural goal to try to find conditions without the above hypotheses on the varieties, in particular, to find conditions necessary and sufficient for interpolation which apply to any given discrete variety. We shall solve this problem in the present note. It turns out, roughly speaking, that a discrete variety V is interpolating for p if and only if V is a subset of the analytic variety $Z(f_1, \cdots, f_m)$ for $m(\geq n)$ functions $f_1, \cdots, f_m \in A_p$ the sum of whose directional derivatives in absolute value is not "too small" (see Theorem 2.5). As a corollary, we solve an open problem posed by Berenstein and Taylor in [**BT3**, p. 10] about interpolation for discrete varieties (see Remark 2.8).

2. Definitions and results. First of all, let us fix the notations which we shall use throughout this paper.

A plurisubharmonic function $p : \mathbf{C}^n \rightarrow [0, \infty)$ is called a weight function if it satisfies the following conditions (c.f. [**BT2**]):

$$\log(1 + |\xi|^2) = O(p(\xi)) \tag{2.1}$$

and there exist constants C_1 and C_2 such that $|\xi - w| \leq 1$ implies

$$p(\xi) \leq C_1 p(w) + C_2. \tag{2.2}$$

Definition 2.1. Let $A(\mathbf{C}^n)$ be the ring of all entire functions on $\mathbf{C}^n$.Then

$$\begin{aligned} A_p &= A_p(\mathbf{C}^n) = \\ &= \{f \in A(\mathbf{C}^n) : |f(\xi)| \leq A \exp(Bp(\xi)) \text{ for some } \quad A,\, B > 0\}. \end{aligned}$$

It is not the specific conditions on p which are important, but rather their consequences for the ring A_p. It follows from (2.1) that $A_p(\mathbf{C}^n)$ contains the polynomials and, from (2.2), that $f \in A_p(\mathbf{C}^n)$ implies $\frac{\partial f}{\partial \xi_k} \in A_p(\mathbf{C}^n)$ (see e.g. [**Ho**]). One can replace (2.2) by the following Hörmander's condition ([**Ho**]): there exist four positive constants $c_1, \cdots, c_4$ such that $|\xi - w| \leq \exp(-c_1 p(w) - c_2)$ implies that $p(\xi) \leq c_3 p(w) + c_4$. We use (2.2) only for the sake of convenience.

Remark 2.2. The two basic examples of such weight functions are $p(\xi) = |\xi|^\rho (\rho > 0)$ and $p(\xi) = |\operatorname{Im} \xi| + \log(1 + |\xi|^2)$ corresponding to the space A_p of all entire functions of order $\leq \rho$ and finite type and the space $\widehat{\mathcal{E}'}(\mathbf{R}^n)$ of Fourier transforms of distributions with compact support in $\mathbf{R}^n$ (see e.g. [**E**]).

Definition 2.3. Let $V = \{\zeta_k\}$ be a discrete variety on $\mathbf{C}^n$, i.e., a discrete sequence in $\mathbf{C}^n$ with $|\zeta_k| \uparrow \infty$ as $k \rightarrow \infty$. Then

$$A_p(V) = \{a = \{a_k\}_{k \in \mathbf{N}} : \exists A,\, B > 0, |a_k| \leq A \exp(Bp(\zeta_k)), \forall k \in \mathbf{N}\}.$$

With the above definitions, the interpolation problem is simply to determine when the following restriction map $\rho : A_p \to A_p(V)$ defined by $\rho(f) = \{f(\zeta_k)\}$, is onto from A_p to $A_p(V)$.

Definition 2.4. A discrete variety $V = \{\zeta_k\}$ is an interpolating variety for $A_p = A_p(\mathbf{C}^n)$ if the restriction map ρ is onto from A_p to $A_p(V)$.

Now we can state our main theorem and its corollaries, whose proofs will be given in the next section.

Theorem 2.5. *Let $V = \{\zeta_k\}$ be a discrete variety on $\mathbf{C}^n$ and $m(\geq n)$ be an integer. Then V is an interpolating variety for $A_p(\mathbf{C}^n)$ if and only if there exist m functions $f_1, f_2, \cdots, f_m \in A_p(\mathbf{C}^n)$ such that*

$$V \subset Z(f_1, f_2, \cdots, f_m) \tag{2.3}$$

and for some ϵ, $C > 0$

$$\sum_{j=1}^{m} |D_u f_j(\zeta_k)| \geq \epsilon \exp(-Cp(\zeta_k)), \forall k \in \mathbf{N},\ u \in \mathbf{S}^{2n-1}, \tag{2.4}$$

where $D_u f(\xi) := \frac{\partial f}{\partial \xi_1} u_1 + \cdots + \frac{\partial f}{\partial \xi_n} u_n$ is the directional derivative of f along the direction $u \in \mathbf{S}^{2n-1}$. Here

$$\mathbf{S}^{2n-1} := \{\xi = (\xi_1, \cdots, \xi_n) : |\xi| := (|\xi_1|^2 + \cdots + |\xi_n|^2)^{\frac{1}{2}} = 1\}.$$

Corollary 2.6. *Let $V = \{\zeta_k\}$ be a discrete subset of $\mathbf{C}^n$ and $m(\geq n)$ be an integer. Then V is an interpolating variety for $A_p(\mathbf{C}^n)$ if and only if there exist m functions $f_1, f_2, \cdots, f_m \in A_p$ such that*

$$V \subset Z(f_1, f_2, \cdots, f_m)$$

and for each $k \in \mathbf{N}$, there exists an $n \times n$ minor $\mathcal{J}$ of the Jacobian matrix of $f_1, \cdots, f_m$ such that

$$|\det \mathcal{J}(\zeta_k)| \geq \epsilon \exp(-Cp(\zeta_k)),$$

where ϵ, C are two positive constants independent of k.

Corollary 2.7. *Let $V = \{\zeta_k\}$ be a discrete subset of $\mathbf{C}^n$. Then V is an interpolating variety for $A_p(\mathbf{C}^n)$ if and only if there exist n functions $f_1, f_2, \cdots, f_n \in A_p$ such that*

$$V \subset Z(f_1, f_2, \cdots, f_n)$$

and for some ϵ, $C > 0$

$$|\det J_{f_1,\cdots,f_n}(\zeta_k)| \geq \epsilon \exp(-Cp(\zeta_k)), \quad k \in \mathbf{N},$$

where $J_{f_1,\cdots,f_n}$ *is the Jacobian matrix of* $f_1, \cdots, f_n$.

Let us mention here that the sufficiency of Corollary 2.6 or 2.7 can not follow from [**BT3**] where the variety was again restricted to be the complete intersection of some functions in A_p. In the case $n = 1$, a stronger version of Corollary 2.7 which allows arbitrary multiplicities can be given. We refer the reader to our papers [**BL1**], [**BL2**] and [**BLV**] for related results in this direction.

Remark 2.8. Observe that when the conditions (2.3) and (2.4) are satisfied for a weight p, they are automatically satisfied for any weight $q \geq p$. This gives an affirmative answer to an open problem in [**BT3**, p. 10] for discrete varieties: Let V be an interpolating variety for $A_p(\mathbf{C}^n)$ and q another weight satisfying $q \geq p$. Is V interpolating for$A_q(\mathbf{C}^n)$?

We conclude this section by providing an interpolation example using Theorem 2.5.

Example 2.9. Let $p_j : \mathbf{C} \to [0, \infty)$ be weights in $\mathbf{C}$ $(1 \leq j \leq n)$ and $V_j = \{z_{k,j}\}_{k=1}^{\infty}$ be an interpolating variety for $A_{p_j}(\mathbf{C})$. Then we claim that

$$V := \{\zeta_k = (\zeta_{k,1}, \cdots, \zeta_{k,n}) : \zeta_{k,j} \in V_j\}_{k=1}^{\infty}$$

is an interpolating variety for $A_p(\mathbf{C}^n)$, where $p(\xi) = p_1(\xi_1) + \cdots + p_n(\xi_n)$ for $\xi = (\xi_1, \cdots, \xi_n)$.

In fact, since V_j is an interpolating variety for A_{p_j}, we know that by Corollary 2.7, there exists an entire function $f_j(z)$ in $A_{p_j}(\mathbf{C})$ such that $V_j \subset Z(f_j)$ and for some ϵ_j, $c_j > 0$,

$$f_j'(\zeta_{k,j}) \geq \epsilon_j \exp(-c_j p_j(\zeta_{k,j})).$$

Let

$$F_j(\xi) = f_j(\xi_j) \quad (1 \leq j \leq n).$$

Then clearly, $F_j \in A_p(\mathbf{C}^n)$ and $V \subset Z(F_1, \cdots, F_n)$. Moreover,

$$|\det J_F(\zeta_k)| \geq \epsilon \exp(-cp(\zeta_k)),$$

with $c = \max c_j$, $\epsilon = \epsilon_1 \cdots \epsilon_n$. Now we conclude by Corollary 2.7 that V is an interpolating variety for $A_p(\mathbf{C}^n)$. Furthermore, $V_1 \times \cdots \times V_n$ is also an interpolation variety. As a concrete example, we see that the lattice

$$\begin{aligned} V &= \mathcal{Z}^2 \times \cdots \times \mathcal{Z}^2 = \\ &= \{(\zeta_1, \cdots, \zeta_n) \in \mathbf{C}^n : \zeta_j \in \mathcal{Z}^2, 1 \leq j \leq n\}, \end{aligned}$$

where $\mathcal{Z}^2 = \{m + in : m, n \in \mathbf{Z}\}$, is an interpolating variety for $A_p(\mathbf{C}^n)$ with $p(\xi) = |\xi|^2 = |\xi_1|^2 + \cdots + |\xi_n|^2$ for $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{C}^n$, since it is known that $\mathcal{Z}^2$ is an interpolating variety for $A_p(\mathbf{C})$ with $p(z) = |z|^2$ for $z \in \mathbf{C}$.

3. Proofs of Theorem 2.5 and its corollaries. Let us first prove the following lemmas.

Lemma 3.1. *Let $\{\zeta_k\}$ be a discrete subset of $\mathbf{C}^n$ and $\delta_k = \inf_{j \neq k}\{|\zeta_j - \zeta_k|\}$. If for some constants ϵ, $B > 0$*

$$\delta_k \geq \epsilon \exp(-Bp(\zeta_k)),\ k \in \mathbf{C} \tag{3.1}$$

then there exists a $M > 0$ such that

$$\sum_{k=1}^{\infty} \exp(-Mp(\zeta_k)) < \infty.$$

Proof: Denote

$$\eta_k = \min\{1, \delta_k\}$$

and

$$\mathcal{B}_k = \mathcal{B}(\zeta_k, \eta_k) := \{\xi \in \mathbf{C}^n : |\xi - \zeta_k| \leq \eta_k\}.$$

Let $dV(\xi)$ be the Euclidean volume element in $\mathbf{C}^n$ and $|\mathcal{B}_k| = \int_{\mathcal{B}_k} dV(\xi)$. Then (3.1) implies that

$$|\mathcal{B}_k| \geq \epsilon_1 \exp(-B_1 p(\zeta_k))$$

for some positive numbers ϵ_1, B_1. We then conclude that for large M,

$$\begin{aligned}
\sum_{k=1}^{\infty} \exp(-Mp(\zeta_k)) &= \sum_{k=1}^{\infty} \frac{1}{|\mathcal{B}_k|} \int_{\mathcal{B}_k} \exp(-Mp(\zeta_k)) dV(\xi) \leq \\
&\leq \frac{1}{\epsilon_1} \sum_{k=1}^{\infty} \int_{\mathcal{B}_k} \exp((B_1 - M)p(\zeta_k)) dV(\xi) \leq \\
&\leq \frac{C_2(B_1 - M)}{\epsilon_1} \sum_{k=1}^{\infty} \int_{\mathcal{B}_k} \exp(C_1(B_1 - M)p(\zeta)) dV(\xi) \leq \\
&\leq \frac{C_2(B_1 - M)}{\epsilon_1} \int_{\mathbf{C}}^{n} \exp(C_1(B_1 - M)p(\zeta)) dV(\xi) < \infty
\end{aligned}$$

by virtue of the property (2.2) and (2.1) of the weight p. ■

Lemma 3.2. *Let $f_1, \cdots, f_m$ be $m(\geq n)$ entire functions in $\mathbf{C}^n$. Then*

$$\sum_{j=1}^{m} |D_u f_j(\zeta_k)| \geq \epsilon \exp(-Cp(\zeta_k)),\ \forall k \in \mathbf{N},\ u \in \mathbf{S}^{2n-1} \tag{3.2}$$

for some constants ϵ, $C > 0$ if and only if for each $k \in \mathbf{N}$, there exists a $n \times n$ minor $\mathcal{K}$ of the Jacobian matrix $J_{f_1 \cdots f_m}$ of $f_1, \cdots, f_m$ such that

$$|\det \mathcal{K}(\zeta_k)| \geq \epsilon_1 \exp(-C_1 p(\zeta_k)), \tag{3.3}$$

where ϵ_1, C_1 are two positive constants independent of k.

Proof: For any matrix $A = (a_{i,j})$, we define $||A|| = \sum_{i,j} |a_{i,j}|$. Then it is easy to verify that

$$||AB|| \leq ||A|| \times ||B||$$

for any matrices A and B provided that the left hand side of the above inequality makes sense.

Suppose that there exist some constants ϵ, $C > 0$ such that (3.2) holds. Denote $J := J_{f_1,\cdots,f_m}$, which is a $m \times n$ matrix-valued function. Then for any $u = (u_1 \cdots u_n) \in \mathbf{S}^{2n-1}$, if $Ju^t = v$, then

$$||v(\zeta_k)|| \geq \epsilon \exp(-Cp(\zeta_k)), \tag{3.4}$$

where u^t denotes the transpose of u. This shows that for each k, the kernel of the mapping $\mathcal{J} = J(\zeta_k) : \mathbf{C}^n \to \mathbf{C}^m$, defined by $\mathcal{J}(u) = J \cdot u^t$ for $u = (u_1, \cdots, u_n)$, is zero and thus the dimension of the image $\mathcal{J}(\mathbf{C}^n)$ is n. Up to an isomorphism, we can identify $\mathcal{J}(\mathbf{C}^n)$ with the space $\mathbf{C}^n$. It is then clear that there exists an operator $T : \mathbf{C}^m \to \mathbf{C}^n$, given by a $n \times m$ matrix such that $Tv^t = v^t$ for $v \in \mathcal{J}(\mathbf{C}^n)$ and $||T|| \leq L$ for some constant $L > 0$. Let $Q = T\mathcal{J}$. Then it is easy to check that the image $Q(\mathbf{C}^n)$ is the whole space $\mathbf{C}^n$. Thus, the matrix Q is invertible. Now set $P = Q^{-1}T$. Then $P\mathcal{J} = (T\mathcal{J})^{-1}T\mathcal{J} = E_n$, the $n \times n$ unit matrix. By the well-known Binet-Cauchy theorem (see e.g. [**A**]), any r-rowed determinant of $P\mathcal{J}$ is equal to a sum of terms each of which is a product of an r-rowed determinant of P and an r-rowed determinant of $\mathcal{J}$. In particular, we have that

$$\sum_{l=1}^{N} (\det P_l)(\det \mathcal{J}_l) = \det E_n = 1, \tag{3.5}$$

where N is an integer only depending on n and m, and P_l and $\mathcal{J}_l$ are $n \times n$ minors of P and $\mathcal{J}$, respectively. On the other hand, for any $u \in \mathbf{S}^{2n-1}$, we have that $Qu^t = T\mathcal{J}u^t = \mathcal{J}u^t$ and so that by (3.4),

$$||Qu^t|| \geq \epsilon \exp(-Cp(\zeta_k)).$$

Notice that $\det Q = \lambda_1 \cdots \lambda_n$, where $\lambda_j' s$ are eigenvalues of Q. Let $u_{\lambda_j} \in \mathbf{S}^{2n-1}$ be the unit eigenvector corresponding to λ_j. Then

$$Qu_{\lambda_j}^t = \lambda_j u_{\lambda_j}.$$

Hence

$$n|\lambda_j| \geq |\lambda_j|||u_{\lambda_j}|| = ||Qu_{\lambda_j}^t|| \geq \epsilon \exp(-Cp(\zeta_k))$$

or

$$|\lambda_j| \geq \frac{1}{n}\epsilon \exp(-Cp(\zeta_k)).$$

We then have that

$$|\det Q(\zeta_k)| \geq \frac{\epsilon^n}{n^n} \exp(-Cnp(\zeta_k)).$$

It is obvious that

$$||Q(\zeta_k)|| = ||T(\zeta_k)\mathcal{J}(\zeta_k)|| \leq ||T(\zeta_k)|| \times ||\mathcal{J}(\zeta_k)|| \leq \\ \leq L||\mathcal{J}(\zeta_k)|| \leq LA \exp(Bp(\zeta_k))$$

for some constants A, $B > 0$ since the $f_1, \cdots, f_m$ are in the space A_p and A_p is closed under differentiation. Thus, $Q^*(\zeta_k) \leq A_1 \exp(B_1 p(\zeta))$ for some constants A_1, $B_1 > 0$, where Q^* denotes the adjoint matrix of Q. It now follows that

$$||Q^{-1}(\zeta_k)|| = ||\frac{Q^*(\zeta_k)}{\det Q(\zeta_k)}|| \leq A_2 \exp(B_2 p(\zeta_k))$$

and thus that

$$||P|| = ||Q^{-1}T|| \leq L||Q^{-1}|| \leq LA_2 \exp(B_2 p(\zeta_k)).$$

Therefore, each $n \times n$ minor P_l $(l = 1, \cdots, N)$ of P satisfies that

$$|\det P_l| \leq A_3 \exp(B_3 p(\zeta_k)).$$

It follows from (3.5) that there exists at least one of l $(l = 1, \cdots, N)$ such that

$$|\det \mathcal{J}_l| \geq \frac{1}{NA_3} \exp(-B_3 p(\zeta_k)).$$

This concludes the proof of the necessity.

Conversely, if (3.3) holds for some $\epsilon_1, C_1 > 0$. For any $u \in \mathbf{S}^{2n-1}$, we let $\mathcal{K}u^t = v$. Then $u^t = \mathcal{K}^{-1}v$ and thus

$$(3.6) \qquad ||u|| \leq ||\mathcal{K}^{-1}|| \times ||v||, \quad \text{or} \quad ||v|| \geq \frac{||u||}{||\mathcal{K}^{-1}||}.$$

Notice that $\mathcal{K}^{-1} = \frac{\mathcal{K}^*}{\det \mathcal{K}}$, where $\mathcal{K}^*$ denotes the adjoint matrix of $\mathcal{K}$. Since A_p is closed under differentiation, we deduce that

$$||\mathcal{K}^*(\xi)|| \leq A_4 \exp(B_4 p(\xi)),\ \xi \in \mathbf{C}^n$$

and thus that by (3.3),

$$||\mathcal{K}^{-1}|| \leq \frac{A_4}{\epsilon_1} \exp((B_4 + C_1)p(\xi)),$$

for some constants A_4, $B_4 > 0$. We then obtain that, by (3.6) and taking into account that $||u|| \geq \frac{1}{\sqrt{n}}$ for any $u \in \mathbf{S}^{2n-1}$,

$$||v|| \geq \epsilon \exp(-Cp(\zeta_k))$$

for some constants ϵ, $C > 0$. Hence

$$\sum_{j=1}^{m} |D_u f_j(\zeta_k)| = ||\mathcal{J}u^t|| \geq ||\mathcal{K}u^t|| \geq \epsilon \exp(-Cp(\zeta_k)).$$

The proof of the lemma is thus complete. ■

We are now going to prove Theorem 2.5. In the sequel, we shall use A and B to denote positive constants the actual values of which may vary from one occurrence to the next.

Proof of Theorem 2.5: Sufficiency: For any fixed $k \in \mathbf{N}$, consider the entire function

$$f_{j,u}(z) := f_j(\zeta_k + uz) : \mathbf{C} \to \mathbf{C}, \quad 1 \leq j \leq m,\ u = (u_1, \cdots, u_n) \in \mathbf{S}^{2n-1}.$$

Then

$$f'_{j,u}(0) = \frac{\partial f_j(\zeta_k)}{\partial \xi_1} u_1 + \frac{\partial f_j(\zeta_k)}{\partial \xi_2} u_2 + \cdots + \frac{\partial f_j(\zeta_k)}{\partial \xi_n} u_n = D_u f_j(\zeta_k).$$

Therefore, by (2.4),

$$\sum_{j=1}^{m} |f'_{j,u}(0)| \geq \epsilon \exp(-Cp(\zeta_k)).$$

It follows then that for any $u \in \mathbf{S}^{2n-1}$ there exists a j_u such that $1 \leq j_u \leq m$ and

$$|f'_{j_u,u}(0)| \geq \frac{1}{m}\epsilon \exp(-Cp(\zeta_k)). \tag{3.7}$$

Denote

$$V_u = \{z \in \mathbf{C} : f_{j_u,u}(z) = 0\}$$

and

$$d_u = \min\{1, dist\{0, V_u \setminus \{0\}\}.$$

Since $f_{j_u} \in A_p$, we have, in $|z| \leq 1$

$$|f_{j_u,u}(z)| = |f_{j_u}(\zeta_k + uz)| \leq A \exp(Bp(\zeta_k + uz)),$$

for some constants A, $B > 0$ independent of u and k. However,

$$|\zeta_k + uz - \zeta_k| = |uz| = |z| \leq 1.$$

Thus by the property (2.2) of p, we deduce that

$$p(\zeta_k + uz) \leq C_1 p(\zeta_k) + C_2,$$

with C_1 and C_2 being the same as in (2.2). This yields that

$$|f_{j_u,u}(z)| \leq A \exp(Bp(\zeta_k)), \quad |z| \leq 1$$

with A, $B > 0$ independent of u, k. Set

$$g_u(z) = \frac{f_{j_u}, u(z)}{z} : \mathbf{C} \to \mathbf{C}.$$

Then on $|z| = 1$ and so in $|z| \leq 1$ by the Maximum Modulus Theorem,

$$|g_u(z)| \leq A \exp(Bp(\zeta_k)). \tag{3.8}$$

Therefore, the function

$$G_u(z) := \frac{g_u(z) - g_u(0)}{3A \exp(Bp(\zeta_k))}$$

satisfies that $G_u(0) = 0$ and $|G_u(z)| < 1$ in $|z| < 1$. By the Schwarz Lemma (see e.g. [**BG**]), $|G_u(z)| \leq |z|$ in $|z| < 1$. In particular, letting $a \neq 0$ be any zero of $f_{j_u,u}$ in $|z| < 1$, then we have, in view of (3.7),

$$\begin{aligned} |a| \geq |G_u(a)| &= |\frac{g_u(0)}{3A \exp(Bp(\zeta_k))}| = \\ &= |\frac{f'_{j_u,u}(0)}{3A \exp(Bp(\zeta_k))}| \geq \epsilon_1 \exp(-c_1 p(\zeta_k)) \end{aligned}$$

for some positive constants ϵ_1, c_1 ($\epsilon_1 < 1$) which are independent of u and k. This implies that

$$d_u \geq \epsilon_1 \exp(-c_1 p(\zeta_k)). \tag{3.9}$$

Using the Carathèodory inequality for $g_u(z)$ (see e.g. [**Le**, p. 19]), we have that in $|z| \leq \frac{d_u}{2}$,

$$\log |\frac{g_u(z)}{g_u(0)}| \geq -\frac{2 \times \frac{d_u}{2}}{d_u - \frac{d_u}{2}} \log(\max_{|z|=d_u} \{|\frac{g_u(z)}{g_u(0)}|\})$$

and so that, by the fact that $g_u(0) = f'_{j_u,u}(0)$, (3.7) and (3.8),

$$|g_u(z)| \geq \epsilon_2 \exp(-c_2 p(\zeta_k)),$$

where ϵ_2, c_2 are positive numbers independent of k and u. Let

$$\hat{d_k} = \epsilon_1 \exp(-c_1 p(\zeta_k)),$$

where ϵ_1 and c_1 are the same as in (3.9). Then for $|z| = \frac{\hat{d_k}}{2}$, we have that

$$|f_{j_u,u}(z)| = |z g_u(z)| \geq \epsilon_3 \exp(-c_3 p(\zeta_k)).$$

Thus, we have proved that for any $u \in \mathbf{S}^{2n-1}$, there exists a j_u with $1 \leq j_u \leq n$ such that for $|z| = \frac{\hat{d_k}}{2}$

$$|f_{j_u}(\zeta_k + uz)| \geq \epsilon \exp(-cp(\zeta_k)).$$

and so that we always have

$$\sum_{j=1}^{m} |f_j(\zeta_k + uz)| \geq \epsilon \exp(-cp(\zeta_k)), \quad u \in \mathbf{S}^{2n-1}, |z| = \frac{\hat{d_k}}{2}, \tag{3.10}$$

for some positive constants c and $\epsilon(\epsilon < 1)$ independent of k and u.

Now for any $k \in \mathbf{N}$, consider the neighborhood

$$U_k := \{\xi \in \mathbf{C}^n : |\xi - \zeta_k| \leq \frac{\hat{d_k}}{2}\}$$

of ζ_k, where $\hat{d_k}$ is defined as above. Note that for any $\xi \in \partial U_k$, we have $\xi = \zeta_k + \frac{\hat{d_k}}{2} u_{\xi-\zeta_k}$ for some unit vector $u_{\xi-\zeta_k} \in \mathbf{S}^{2n-1}$. Hence by (3.10), we have that for $\xi \in \partial U_k$,

$$\sum_{j=1}^{m} |f_j(\xi)| = \sum_{j=1}^{m} |f_j(\zeta_k + \frac{\hat{d_k}}{2} u_{\xi-\zeta_k})| \geq \epsilon \exp(-cp(\zeta_k)). \tag{3.11}$$

Let

$$S(f_1, f_2, \cdots, f_m; \epsilon, c) = \{\xi \in \mathbf{C}^n : \sum_{j=1}^{m} |f_j(\xi)| < \epsilon \exp(-cp(\xi))\}.$$

The component of $S(f_1, f_2, \cdots, f_m; \epsilon, c)$ containing ζ_k is denoted by $\hat{V}_k$. Then obviously, $\hat{V}_k \subset U_k$ by (3.11). We claim that U_k does not contain any ζ_j if $j \neq k$. In fact, letting $d'_j = |\zeta_j - \zeta_k|$, we know that $\zeta_j = \zeta_k + d'_j u_{\zeta_j - \zeta_k}$ for some unit vector $u_{\zeta_j - \zeta_k} \in \mathbf{S}^{2n-1}$. By the definition of d_u and (3.9), we deduce that

$$d'_j \geq d_{u_{\zeta_j - \zeta_k}} \geq \epsilon_1 \exp(-c_1 p(\zeta_k)) = \hat{d}_k.$$

This implies that $\zeta_j \notin U_k$.

Thus for any sequence $\{a_k\} \in A_p(V)$, we can define an analytic function $F(\xi) : S(f_1, \cdots, f_m; \epsilon, c) \mapsto \mathbf{C}$ by:

$$F(z) = \begin{cases} a_k, & \text{if } z \in \hat{V}_k; \\ 0, & \text{if } z \in S(f_1, \cdots, f_m; \epsilon, c) \setminus \cup_{k \in \mathbf{N}} \hat{V}_k. \end{cases}$$

Then the sufficiency follows from the following semi-local to global extension theorem ([**BT2**]): suppose that for some ϵ, $c > 0$ λ is a function analytic on $S(f_1, \cdots, f_m; \epsilon, c)$ such that

$$|\lambda(\xi)| \leq A_1 \exp(B_1 p(\xi))$$

for all $\xi \in S(f_1, \cdots, f_m; \epsilon, c)$. Then there exists an entire function $f \in A_p$, constants ϵ_1, c_1, A, $B > 0$ such that for $\xi \in S(f_1, \cdots, f_m; \epsilon, c)$,

$$f(\xi) - \lambda(\xi) = (g_1 f_1 + \cdots + g_m f_m)(\xi)$$

and $|g_j(\xi)| \leq A \exp(Bp(\xi))$ for all $\xi \in S(f_1, \cdots, f_m; \epsilon_1, c_1)$. In particular, $f(\xi) = \lambda(\xi)$ on $Z(f_1, \cdots, f_m)$. Applying this result with $\lambda(\xi) = F(\xi)$, we then obtain an entire function $f \in A_p(\mathbf{C}^n)$ such that $f(\zeta_k) = F(\zeta_k) = a_k$. This completes the proof of sufficiency.

Necessity: For any $M > 0$, we set

$$E = \{a = \{a_k\}_{k \in \mathbf{N}} : |a_k| \exp(-Mp(\zeta_k)) \leq 1, k \in \mathbf{N}\}.$$

Then the space E is complete under the metric induced by the norm

$$||a||_E := \sup\{|a_k| \exp(-Mp(\zeta_k)) : k \in \mathbf{N}\}.$$

For $l \in \mathbf{N}$, let

$$E_l = \{f \in \mathbf{A}_p : |f(\xi)| \leq l \exp(lp(\xi))\}.$$

By the hypothesis that $\{\zeta_k\}$ is an interpolating variety for $A_p(\mathbf{C}^n)$, we see that $E = \cup_{l=1}^{\infty} \hat{E}_l$, where $\hat{E}_l := \{\{f(\zeta_k)\} \in E : f \in E_l\}$.

We next prove that each $\hat{E}_l$ is closed subset of E. In fact, suppose that f_j is a sequence of functions in E_l satisfying that for each j, $\{f_j(\zeta_k)\} \in E$ and

$$\{f_j(\zeta_k)\} \to a \in E, \quad \text{as} \quad j \to \infty.$$

Since $|f_j(\xi)| \leq l \exp(l(p(\xi))$ for any $j \in \mathbf{N}$ and the weight p has the property (2.2), $\{f_j\}$ is locally bounded. By the $\mathbf{C}^n$ version of Montel's theorem (see e.g. [**G**, p. 54]), $\{f_j\}$ is a normal family. By passing to subsequence, we can assume, without loss of generality, that $f_j \to f$ normally. Now, it follows from the Weierstrass theorem that f is an entire function on $\mathbf{C}^n$. It is clear that $f \in E_l$ and $\{f(\zeta_k)\} = a$ by the uniqueness of limit. Thus $a \in \hat{E}_l$. That is, $\hat{E}_l$ is closed. We conclude by the the Baire-category theorem (see e.g. [**La**]) that some $\hat{E}_l$ has non-empty interior. It is no loss of generality to assume that

$$\hat{E}_l \supset \{a \in E : ||a||_E < \epsilon\}$$

for some $\epsilon > 0$, from which we readily obtain a sequence $\{h_k\}$ with $h_k \in E_l$ such that

$$h_k(\zeta_k) = \epsilon \exp(Mp(\zeta_k)) \quad \text{and} \quad h_k(\zeta_j) = 0, j \neq k. \tag{3.12}$$

Recall that $\mathbf{S}^{2n-1} = \{u = (u_1, \cdots, u_n) \in \mathbf{C}^n : |u|^2 = (|u_1|^2 + \cdots + |u_n|^2)^{\frac{1}{2}} = 1\}$. We can cover $\mathbf{S}^{2n-1}$ by its n subsets

$$\Delta_j := \{u = (u_1, \cdots, u_n) \in \mathbf{S}^{2n-1} : |u_j| \geq \frac{1}{\sqrt{n}}\}$$

$(1 \leq j \leq n)$. That is, $\mathbf{S}^{2n-1} = \cup_{j=1}^{n} \Delta_j$. For each fixed $j(1 \leq j \leq n)$, we define for $\xi \in \mathbf{C}^n$,

$$f_j(\xi) = \sum_{k=1}^{\infty} (\xi_j - \zeta_{k,j}) h_k^2(\xi) / \exp(2Mp(\zeta_k)), \quad 1 \leq j \leq n, \tag{3.13}$$

where $\xi = (\xi_1, \cdots, \xi_n)$ and $\zeta_k = (\zeta_{k,1}, \cdots, \zeta_{k,n})$. We shall prove that for each j, the above series converges to an entire function f_j in $A_p(\mathbf{C}^n)$. Assuming this for the moment, we see that

$$V \subset Z(f_1, f_2, \cdots, f_n) \tag{3.14}$$

and f_j can be expanded into the following power series at each ζ_k,

$$f_j(\xi) = \epsilon^2(\xi_j - \zeta_{k,j}) + \\ + \sum_{i_1+\cdots+i_n=1}^{\infty} C_{i_1,\cdots,i_n}(\xi_1 - \zeta_{k,1})^{i_1}\cdots(\xi_j - \zeta_{k,j})^{i_j+1}\cdots(\xi_n - \zeta_{k,n})^{i_n}.$$

From this, it follows that

$$\nabla f_j(\zeta_k) := (\frac{\partial f_j}{\partial \xi_1}(\zeta_k), \cdots, \frac{\partial f_j}{\partial \xi_n}(\zeta_k)) = (0, \cdots, 0, \epsilon^2, 0, \cdots, 0) \tag{3.15}$$

with the j-th entry being ϵ^2. On the other hand, for any $u \in \mathbf{S}^{2n-1}$, there exists a $j(1 \le j \le n)$ such that $u \in \Delta_j$ and so that $|u_j| \ge \frac{1}{\sqrt{n}}$. Hence by (3.15)

$$\sum_{i=1}^{n} |D_u f_i(\zeta_k)| \ge |D_u f_j(\zeta_k)| =$$

$$= |u_1 \frac{\partial f_j}{\partial \xi_1}(\zeta_k) + \cdots + u_j \frac{\partial f_j}{\partial \xi_j}(\zeta_k) + \cdots + u_n \frac{\partial f_j}{\partial \xi_n}(\zeta_k)| =$$

$$= |u_j \frac{\partial f_j}{\partial \xi_j}(\zeta_k)| = \epsilon^2 |u_j| \ge \epsilon^2 \frac{1}{\sqrt{n}}. \tag{3.16}$$

Thus, the necessity will follow from (3.14) and (3.16) once we show that each $f_j \in A_p$ for $1 \le j \le n$ (if $m > n$, we can easily add $m - n$ functions $f_{n+1}, \cdots f_m$ such that (2.3) and (2.4) hold) . In fact, since $h_k \in E_l$ for any $k \in \mathbf{N}$, we have for any $\xi \in \mathbf{C}^n$

$$|h_k(\xi)| \le l \exp(lp(\xi)).$$

Also,

$$|\xi_j - \zeta_{k,j}| \le |\xi - \zeta_k| \le |\xi| + |\zeta_k| \le \\ \le A \exp(Bp(\xi)) + A \exp(Bp(\zeta_k))$$

for some positive constants A, B by the property (2.1) of the weight p. Thus, for any $k \in \mathbf{N}$

$$|(\xi_j - \zeta_{k,j}) h_k^2(\xi) / \exp(2Mp(\zeta_k))| \le A \exp(Bp(\xi)) \exp((C - 2M)p(\zeta_k)), \tag{3.17}$$

for some constants A, B, $C > 0$ with C being independent of M. We claim that

$$\sum_{k=1}^{\infty} \exp((C - 2M)p(\zeta_k)) < \infty,$$

provided that M is large. To this end, let

$$\delta_k = \inf_{j \neq k}\{|\zeta_j - \zeta_k|\}, \quad \eta_k = \min\{1, \delta_k\},$$

and

$$\mathcal{B}_k = \mathcal{B}(\zeta_k, \eta_k) := \{\xi \in \mathbf{C}^n : |\xi - \zeta_k| \leq \eta_k\}.$$

By repeating the proof of (3.12) for $M = 1$, one can obtain another sequence $\{g_k\}$ of entire functions satisfying that

$$g_k(\zeta_k) = \epsilon_1, \quad g_k(\zeta_j) = 0,\, j \neq k, \tag{3.18}$$

where ϵ_1 is a positive number, and for $\xi \in \mathbf{C}^n$, $k \in \mathbf{N}$

$$|g_k(\xi)| \leq A \exp(Bp(\xi)) \tag{3.19}$$

for some positive numbers A and B. Denote $G_k(\xi) = g_k(\xi) - g_k(\zeta_k)$. Then $G_k(\zeta_k) = 0$ and for $|\xi - \zeta_k| \leq 1$

$$|G_k(\xi)| \leq |g_k(\xi)| + |g_k(\zeta_k)| \leq A_1 \exp(B_1 p(\zeta_k))$$

for some constants A_1, $B_1 > 0$, in view of (3.19) and the property (2.2) of p. Recall the following $\mathbf{C}^n$ version of Schwarz's Lemma (c.f. [**G**, p. 7]): If F is holomorphic in an open neighborhood of a closed ball $\bar{\mathcal{B}}(\zeta, r)$ and $|F(\xi)| \leq \alpha$ for all $\xi \in \bar{\mathcal{B}}(\zeta, r)$, and if $\frac{\partial^{|I|} F}{\partial \xi^I}(\xi) = 0$ whenever $|I| < m$ for some $m \in \mathbf{N}$, where $I = (i_1, i_2, \cdots, i_n)$ is a multi-index, $|I| = i_1 + \cdots + i_n$, then

$$|F(\xi)| \leq \alpha r^{-m} |\xi - \zeta|^m, \quad \xi \in \bar{\mathcal{B}}(\zeta, r).$$

Applying this result to the function $G_k(\xi)$ with $m = 1$, we have that for $\xi \in \bar{\mathcal{B}}(\zeta_k, 1)$,

$$|G_(\xi)| \leq A_1 \exp(B_1 p(\zeta_k)) |\xi - \zeta_k|,$$

for some positive constants A_1, B_1. In particular, if $\zeta_j \in \bar{\mathcal{B}}(\zeta_k, 1), j \neq k$, then we have that, in view of (3.18),

$$\epsilon_1 = |g_k(\zeta_k)| = |G_k(\zeta_j)| \leq A_1 \exp(B_1 p(\zeta_k)) |\zeta_j - \zeta_k|$$

or

$$|\zeta_j - \zeta_k| \geq \epsilon_1 A_1^{-1} \exp(-B_1(\zeta_k)).$$

This shows that

$$\delta_k \geq \epsilon_1 A_1^{-1} \exp(-B_1 p(\zeta_k)).$$

It now follows from Lemma 3.1 that

$$\sum_{k=1}^{\infty} \exp((C-2M)p(\zeta_k)) := D < \infty,$$

provided that M is large. Combining this result with (3.17), we know that the series (3.13) is uniformly convergent in compact sets of $\mathbf{C}^n$ and so f_j is an entire function on $\mathbf{C}^n$. Furthermore,

$$|f_j(\xi)| \leq A\exp(Bp(\xi)) \times D,$$

that is, $f_j \in A_p(\mathbf{C}^n)$.

The proof of Theorem 2.5 is thus complete. ■

Proof of Corollary 2.6: The corollary follows directly from Theorem 2.5 and Lemma 3.2. ■

Proof of Corollary 2.7: The corollary is obtained from Corollary 2.6 by taking $m = n$. ■

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